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THE RELATIONSHIP BETWEEN THE POWERS OF AN INVERTIBLE MATRIX AND THOSE OF ITS INVERSE

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Abstract. In the current paper, we establish the relationship between the powers of an invertible matrix and the powers of its inverse. More precisely, we prove that if A is an invertible matrix and, if $A^n = (A_{i,j}(n))$ for all positive integer n, then $A^{-n} = (A_{i,j}(-n))$.

1. Introduction

Different kinds of matrices play very important roles in several scientific works, where the integral powers of these matrices are often used.
There are two common methods for finding positive integer powers of square matrices. The first method uses the diagonal form for diagonalizable matrices and the Jordan normal form for non-diagonalizable matrices. The second method consists in using the Cayley-Hamilton theorem. But if we want to determine integer powers of a square invertible matrix A, we compute the positive powers A^n, we also compute A^{-1} and we raise A^{-1} to the positive integers n or else to prove by induction on n that the formula obtained for positive powers A^n is still true when n is negative.
In the case where A is a 2×2 complex matrix, the study of a suitable recursive sequence allows us to obtain an explicit form of the elements of A^n for all positive integers n (see [2]). It can be easily verified when A is invertible that if we replace n by $-n$ in the elements of A^n obtained, we get the matrix A^{-n}.

In this paper, we prove that the same result holds for any invertible complex matrix A. We shall give the proof using only combinatorial relations and the Jordan form.

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2. Proof of the result

Let A be any square matrix, and let $A(n)$ denote the sequence of matrices defined by $A(n) = (A_{i,j}(n))_{i,j}$ for all positive integers n, where $A_{i,j}(n)$ are the elements of A^n. From the definition of matrix multiplication, the sequence $A(-n) = (A_{i,j}(-n))_{i,j}$ is also well-defined for all $n \in \mathbb{N}$. Of course, if $A_{i,j}(n) = c$ is a constant sequence for a given i, j, then $A_{i,j}(-n) = c$.

THEOREM 2.1. *Let A be a complex square invertible matrix of order p. Then $A^{-1}(n) = A(-n)$ for all positive integers n.*

Proof. Let A be a complex square matrix with order p. It is well known from Linear Algebra that A has the following decomposition

$$A = T^{-1}JT,$$

where J is a Jordan matrix, and T is some non-singular matrix. The Jordan matrix J has the form

$$J = \begin{pmatrix} J_{p_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{p_k}(\lambda_k) \end{pmatrix},$$

where

$$J_{p_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$$

is a Jordan block of size p_i with eigenvalue λ_i.
Since $A(n) = T^{-1}J(n)T$ and

$$J(n) = \begin{pmatrix} J_{p_1}(\lambda_1)^n & & \\ & \ddots & \\ & & J_{p_k}(\lambda_k)^n \end{pmatrix}$$

for all positive integers n, then we may assume without any loss of generality that $A = J_p(\lambda)$ is a Jordan block.

Newton's binomial formula gives

$$A(n)=\begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \cdots & \binom{n}{p-1}\lambda^{n-p+1} \\ 0 & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \binom{n}{1}\lambda^{n-1} \\ 0 & \cdots & 0 & \lambda^n \end{pmatrix}=(A_{i,j}(n)),$$

where $\binom{n}{k}=\dfrac{n!}{k!(n-k)!}$, with the convention that $\binom{n}{k}=0$ for $k>n$, and

$$A_{i,j}(n)=\begin{cases} 0 & \text{if} \quad j<i \\ \lambda^n & \text{if} \quad i=j \\ \binom{n}{j-i}\lambda^{n-(j-i)} & \text{if} \quad i<j \end{cases}$$

If we replace n by $-n$ in this matrix, we obtain the following one

$$A(-n)=\begin{pmatrix} \lambda^{-n} & \binom{-n}{1}\lambda^{-n-1} & \cdots & \binom{-n}{p-1}\lambda^{-n-p+1} \\ 0 & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \binom{-n}{1}\lambda^{-n-1} \\ 0 & \cdots & 0 & \lambda^{-n} \end{pmatrix},$$

where

$$A(-n)_{i,j}=\begin{cases} 0 & \text{if} \quad j<i \\ \lambda^{-n} & \text{if} \quad j=i \\ \binom{-n}{j-i}\lambda^{-n-(j-i)} & \text{if} \quad i<j \end{cases}$$

and

$$\binom{-n}{k}=\begin{cases} \dfrac{-n(-n-1)\dots(-n-k+1)}{k!} & \text{if} \quad k\neq 0 \\ 1 & \text{if} \quad k=0 \\ 0 & \text{if} \quad k>n. \end{cases}$$

If we set $A(-n)A(n)=(D_{i,j})$, then it is clear that $D_{i,i}=1$ and $D_{i,j}=0$ if $1\leq j<i\leq p$, so it remains to show that $D_{i,j}=0$ if $1\leq i<j\leq p$.

Let (i,j) be a pair of integers with $1 \le i < j \le p$. Then we have

$$\begin{aligned}
D_{i,j} &= \sum_{k=1}^{p} A(-n)_{i,k} A_{k,j}(n) \\
&= \sum_{k=i}^{j} A(-n)_{i,k} A_{k,j}(n) \\
&= \sum_{k=i}^{j} \binom{-n}{k-i} \lambda^{-n-(k-i)} \binom{n}{j-k} \lambda^{n-(j-k)} \\
&= \lambda^{i-j} \sum_{k=i}^{j} \binom{-n}{k-i} \binom{n}{j-k} \\
&= \lambda^{i-j} \sum_{k=0}^{j-i} \binom{-n}{k} \binom{n}{j-i-k}.
\end{aligned}$$

So it is enough to show that for every integer m with $1 \le m \le n$, we have the following formula

$$\sum_{k=0}^{m} \binom{-n}{k} \binom{n}{m-k} = 0.$$

Since

$$\begin{aligned}
\binom{-n}{k} &= (-1)^k \frac{(n+k-1)\cdots n}{k!} \\
&= (-1)^k \binom{n+k-1}{n-1},
\end{aligned}$$

then

$$\begin{aligned}
\sum_{k=0}^{m} \binom{-n}{k} \binom{n}{m-k} &= \sum_{k=0}^{m} (-1)^k \binom{n+k-1}{n-1} \binom{n}{m-k} \\
&= n \sum_{k=0}^{m} \frac{(-1)^k}{n+k} \binom{n+k}{m} \binom{m}{k}.
\end{aligned}$$

To show that $\displaystyle\sum_{k=0}^{m} \frac{(-1)^k}{n+k} \binom{n+k}{m} \binom{m}{k} = 0$, we consider the following polynomial

$$P(X) = X^m (X+1)^{n-1}.$$

By using the Newton's binomial formula, we find that

$$\begin{aligned} P(X) &= (X+1-1)^m (X+1)^{n-1} \\ &= (X+1)^{n-1}(\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(X+1)^k) \\ &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(X+1)^{n+k-1}. \end{aligned}$$

So we have shown that $P(X)$ can be written in two different forms. We are now going to compute the coefficient of the monomial X^m in the primitive of $P(X)$ using these two forms.
The second form of $P(X)$ gives

$$\begin{aligned} \int P(X) &= \sum_{k=0}^{m}\frac{(-1)^{m-k}}{n+k}\binom{m}{k}(X+1)^{n+k} \\ &= \sum_{k=0}^{m}\frac{(-1)^{m-k}}{n+k}\binom{m}{k}(\sum_{i=0}^{n+k}\binom{n+k}{i}X^i). \end{aligned}$$

Since $m \leq n$, then $m \leq n+k$ for all $k \geq 0$. Therefore X^m appears in $\sum_{i=0}^{n+k}\binom{n+k}{i}X^i$ for each $k \geq 0$. Then the coefficient of X^m in $\displaystyle\int P(X)$ is equal to

$$\sum_{k=0}^{m}\frac{(-1)^{m-k}}{n+k}\binom{m}{k}\binom{n+k}{m}.$$

Now since

$$\begin{aligned} P(X) &= X^m(\sum_{k=0}^{n-1}\binom{n-1}{k}X^k) \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}X^{k+m}, \end{aligned}$$

then

$$\int P(X) = \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{1}{k+m+1}X^{k+m+1},$$

which means that the coefficient of X^m in $\displaystyle\int P(X)$ is 0. This proves that

$\displaystyle\sum_{k=0}^{m} \frac{(-1)^{m-k}}{n+k}\binom{m}{k}\binom{n+k}{m} = 0$, and hence $\displaystyle\sum_{k=0}^{m} \frac{(-1)^{k}}{n+k}\binom{m}{k}\binom{n+k}{m} =$ 0.
This completes the proof of the theorem. □

REMARK 2.2. If A is an involutory matrix, the sequence $A(n)$ is even.

EXAMPLE 2.3. Let $A = (a_{i,j})$ be the invertible matrix of even order m, defined by

$$\begin{cases} a_{i,i+1} = a_{i+1,i} = 1 & \text{for} \quad i = 1, 3, 4, \dots, m-1 \\ a_{i,i} = 2 & \text{for} \quad i = 2, 4, 6, \dots, m \\ 0, & \qquad \text{otherwise.} \end{cases}$$

In [1], it was shown that $A(n) = A^n = (c_{i,j})$ where

$$\begin{cases} c_{i-1,i-1} & = & \frac{1}{\alpha-\beta}[\beta^n\alpha - \beta\alpha^n] \\ c_{i,i} & = & \frac{1}{\alpha-\beta}[\alpha^{n+1} - \beta^{n+1}] \\ c_{i-1,i} & = & \frac{1}{\alpha-\beta}[\alpha^n - \beta^n] \\ c_{i,i-1} & = & c_{i-1,i} \\ 0, & & \text{otherwise} \end{cases}$$

$i = 2, 4, 6, \dots, m$, $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.
Replacing n by $-n$ in the above entries, one obtains

$$A(-n) = \begin{cases} b_{i-1,i-1} & = & \frac{1}{\alpha-\beta}[\beta^{-n}\alpha - \beta\alpha^{-n}] \\ b_{i,i} & = & \frac{1}{\alpha-\beta}[\alpha^{-n+1} - \beta^{-n+1}] \\ b_{i-1,i} & = & \frac{1}{\alpha-\beta}[\alpha^{-n} - \beta^{-n}] \\ b_{i,i-1} & = & b_{i-1,i} \\ 0, & & \text{otherwise} \end{cases}$$

where $i = 2, 4, 6, \dots, m$.
Now if we take $n = 1$, we find that

$$A(-1) = \begin{cases} b_{i-1,i-1} & = & -(\alpha+\beta) = -2 \\ b_{i,i-1} & = & b_{i-1,i} = 1 \\ 0, & & \text{otherwise} \end{cases}$$

where we have used the fact that $\alpha\beta = -1$. One can easily verify that $A(-1)A = I_m$, which shows that A is invertible and $A^{-1} = A(-1)$. Then, by using the theorem, one gets $A^{-n} = A(-n)$. Now if we replace α^{-n} by $(-\beta)^n$ and β^{-n} by $(-\alpha)^n$, we obtain exactly the same result as in [1].

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